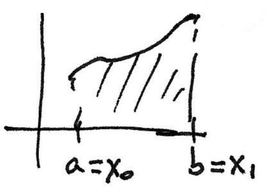
4.3 Elements of Numerical Integration

Trapezoidal rule:

For $f(x)$ conts on $[a,b]$, consider an approx. using Lagrange Polyn. of degree 1, $P_1(x)$.



$$f(x) = P_1(x) + \text{Error} = P_1(x) + \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1) \quad (*)$$

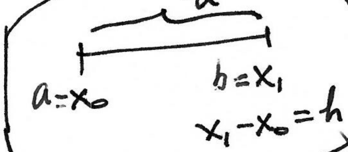
$$\Rightarrow \quad f(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) + (\text{Error})$$

$$\Rightarrow \int_{a=x_0}^{b=x_1} f(x)\,dx = \overbrace{\int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} f(x_0)\,dx}^{\equiv I_0} + \overbrace{\int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} f(x_1)\,dx}^{\equiv I_1} +$$

$$+ \overbrace{\int_{x_0}^{x_1} \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1)\,dx}^{\equiv \mathcal{E}}$$

Now, $$I_0 = \int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} f(x_0)\,dx = \frac{f(x_0)}{x_0-x_1} \left.\frac{(x-x_1)^2}{2}\right|_{x_0}^{x_1} = \frac{f(x_0)}{x_0-x_1}\left(-\frac{(x_0-x_1)^2}{2}\right)$$

$$= -\frac{f(x_0)(x_0-x_1)}{2} = +\frac{h}{2} f(x_0)$$



$$I_1 = \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} f(x_1)\,dx = \frac{f(x_1)}{x_1-x_0} \left.\frac{(x-x_0)^2}{2}\right|_{x_0}^{x_1} = \frac{f(x_1)(x_1-x_0)}{2} = \frac{h}{2} f(x_1)$$

$$\boxed{h = b-a}$$

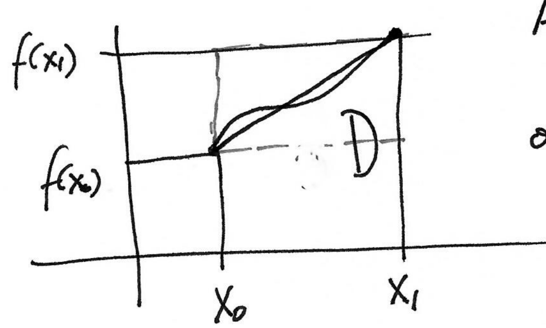
$(*)$ $$\boxed{f(x) \simeq P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)}$$

Therefore,

$$\int_{x_0}^{x_1} f(x)\,dx \simeq I_0 + I_1 = \frac{h}{2}\left(f(x_0)+f(x_1)\right)$$

Trapezoidal rule

Geometric Derivation



$$\text{Area}(D) \simeq \frac{1}{2}\left(f(x_1)+f(x_0)\right)(x_1-x_0)$$

or $$\int_{x_0}^{x_1} f(x)\,dx \simeq \frac{1}{2}\left(f(x_1)+f(x_0)\right)(x_1-x_0)$$

Error

$$\frac{1}{2}\int_{x_0}^{x_1} f''(\xi(x))\,\overbrace{(x-x_0)(x-x_1)}^{<0}\,dx =$$

Weighted MVT Int

$$= \frac{1}{2} f''(\xi)\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx = \frac{f''(\xi)}{2}\left[\frac{x^3}{3}-(x_0+x_1)\frac{x^2}{2}+x_0x_1x\right]_{x_0}^{x_1} =$$

$$= \frac{f''(\xi)}{2}\left(-\frac{1}{6}\right)(x_1-x_0)^3 = -\frac{f''(\xi)}{2}\frac{h^3}{6} = -f''(\xi)\frac{h^3}{12}$$

Remark: Weighted MVT for integrals $g(x)$ same sign on $[a,b]$.

$$\int_a^b \underset{\text{cont}}{f(x)}\,\underset{\text{integr}}{g(x)}\,dx = f(c)\int_a^b g(x)\,dx, \quad \text{for } c\in(a,b)$$

$$\frac{x_1^3}{3} - (x_0+x_1)\frac{x_1^2}{2} + x_0 x_1^2$$

$$-\left(\frac{x_0^3}{3} - (x_0+x_1)\frac{x_0^2}{2} + x_0^2 x_1\right)$$

$$= \frac{x_1^3}{3} - \frac{x_1^3}{2} - \frac{x_0 x_1^2}{2} + x_0 x_1^2$$

$$-\left(\frac{x_0^3}{3} - \frac{x_0^3}{2} - x_1\frac{x_0^2}{2} + x_0^2 x_1\right)$$

$$= \frac{2x_1^3 - 3x_1^3}{6} + \frac{x_0 x_1^2}{2} + \frac{x_0^3}{6} - \frac{x_0^2 x_1}{2}$$

$$= -\frac{x_1^3}{6} + \frac{x_0^3}{6} + \frac{x_0 x_1}{2}(x_1 - x_0)$$

$$= -\frac{1}{6}\left(x_1^3 - x_0^3 - 3x_0x_1^2 + 3x_0^2x_1\right)$$

$$(x_1-x_0)^3 = (x_1-x_0)^2(x_1-x_0) = \left(x_1^2 - 2x_1x_0 + x_0^2\right)(x_1-x_0)$$

$$= x_1^3 - 2x_1^2x_0 + 2x_1x_0^2 - x_0^3$$

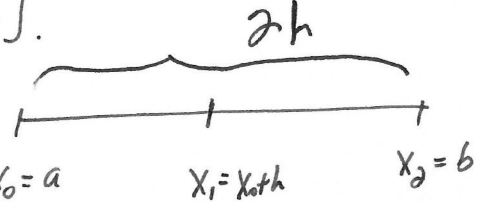
$$\quad - x_1^2x_0 + x_0^2x_1$$

$$= x_1^3 + 3x_1x_0^2 - 3x_1^2x_0 - x_0^3$$

$$= (x_1-x_0)^3$$

Simpson's Rule: (Using Taylor's expansion)

$f \in C^4[a,b]$. $\boxed{h \equiv \frac{b-a}{2}}$



$f(x)$ expanded in the third Taylor polynomial about x_1.

$$f(x) = f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 + \frac{f'''(x_1)}{3!}(x-x_1)^3 + \frac{f^{(IV)}(\xi(x))}{4!}(x-x_1)^4$$

Then,

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} f(x_1)\,dx + \int_{x_0}^{x_2} f'(x_1)(x-x_1)\,dx + \cdots + \underbrace{\frac{1}{4!}\int_{x_0}^{x_2} f^{(IV)}(\xi(x))(x-x_1)^4\,dx}_{=\text{ Error 1}}$$

$$= f(x_1)\underbrace{(x_2-x_0)}_{=2h} + f'(x_1)\left.\frac{(x-x_1)^2}{2}\right|_{x_0}^{x_2} + \frac{f''(x_1)}{2}\left.\frac{(x-x_1)^3}{3}\right|_{x_0}^{x_2} + \frac{f'''(x_1)}{3!}\cancelto{0}{\left.\frac{(x-x_1)^4}{4}\right|_{x_0}^{x_2}} + \text{Error 1}.$$

$$= 2hf(x_1) + f'(x_1)\cancelto{0}{\left(\frac{\overbrace{(x_2-x_1)^2}^{h^2}}{2} - \frac{\overbrace{(x_0-x_1)^2}^{h^2}}{2}\right)} + \frac{f''(x_1)}{2}\left(\frac{\overbrace{(x_2-x_1)^3}^{h^3}}{3} - \frac{\overbrace{(x_0-x_1)^3}^{(-h)^3}}{3}\right)$$

$$+ \cdots$$

$$\therefore \int_{x_0}^{x_2} f(x)\,dx = 2h\, f(x_1) + \frac{f''(x_1)}{\cancel{2}} \frac{\cancel{2} h^3}{3} + \text{Error } 1.$$

Now, we approximate $f''(x_1)$ using centered differences

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\, f(x_1) + \frac{h^{\cancel{3}}}{3}\left[\frac{f(\overset{=x_0}{x_1-h}) - 2f(x_1) + f(\overset{=x_2}{x_1+h})}{\cancel{h^2}}\right]$$

$$+ \underbrace{\frac{h^3}{3}\left(-\frac{h^2}{12} f^{(IV)}(\xi_2(x))\right)}_{\hookrightarrow = \text{Error } 2.} + \text{Error } 1$$

$$\Rightarrow \int_{x_0}^{x_2} f(x)\,dx = h\Big(\underbrace{2f(x_1) - \tfrac{2}{3} f(x_1)}_{4/3\, f(x_1)}\Big) + \frac{h}{3}\left(f(\overset{=x_0}{x_1-h}) + f(\overset{=x_2}{x_1+h})\right)$$

$$+ \text{Error } 1 + \text{Error } 2.$$

or $$\boxed{\int_{a=x_0}^{b=x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \text{Err. } 1 + \text{Err } 2.}$$

$$\text{Error } 1 = \frac{1}{4!}\int_{x_0}^{x_2} f^{(4)}(\xi(x))\,(x-x_1)^4\,dx =$$

I MVT weighted

$$= \frac{1}{4!} f^{(4)}(\xi_1) \int_{x_0}^{x_2} (x-x_1)^4\,dx = \frac{1}{4!} f^{(4)}(\xi_1)\left(\frac{(x-x_1)^5}{5}\right)\Bigg|_{x_0}^{x_2} =$$

$$\frac{1}{4!} f^{(4)}(\xi_1)\,\frac{2h^5}{5} = \frac{h^5}{60} f^{(4)}(\xi_1)$$

$$\therefore \boxed{\text{Error } 1 = \frac{h^5}{60} f^{(4)}(\xi_1),} \quad \xi_1 \in (a,b).$$

$$\text{Error } 2 = \frac{h^3}{3}\left(-\frac{h^2}{12} f^{(IV)}\left(\xi_2(x_1)\right)\right) = \frac{-h^5}{36} f^{(IV)}(\xi_2)$$

$$\therefore \text{Error} = \text{Error } 1 + \text{Error } 2 = -\frac{h^5}{12}\left[\frac{1}{3} f^{(IV)}(\xi_2) - \frac{1}{5} f^{(IV)}(\xi_1)\right]$$

$a < \xi_1, \xi_2 < b$.

It can be shown using an alternative method (Ex. 18.)

that

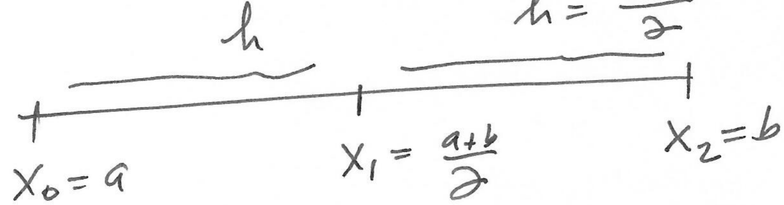
$$\text{Error} = -\frac{h^5}{90} f^{(4)}(\hat{\xi}), \qquad \hat{\xi} \in (a,b).$$

Summarizing Simpson's rule is defined as $\left(f \in C^4_{(a,b)}\right)$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{h^5}{90} f^{(4)}(\hat{\xi}).}$$

or $$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] +$$

$h = \frac{b-a}{2}$



Observe that Simpson's rule is exact for polynomial degree 3.

or $$= \frac{h}{3}\left[f(a) + f(a+h) + f(b)\right] + \text{Error}.$$

Def.- (Degree of accuracy or precision)

The DA of a Quadrature integration formula is the largest positive "n" such that the formula is exact, for X^K each $K = 0, 1, .., n$.

Application:

a) Since error of Trapezoidal rule $= -\frac{h^3}{12} f''(\xi)$

if $f(x) = x \Rightarrow f''(x) \equiv 0 \Rightarrow$ Error Trap. rule $= 0$.

b) Since error of Simpson's rule $= -\frac{h^5}{90} f^{(IV)}(\xi)$

then $f(x) = x^3$ is s.t. $f^{(IV)}(x) = 0$

but $f(x) = x^4$ is s.t. $f^{(IV)}(x) = 12 \neq 0$

Therefore,

a₁) Degree of precision Trapezoidal rule $= 1$

b₁) " " " Simpson's rule $= 3$.

Theorem.- (Exercise #25)

A quadrature formula has degree of precision "n" if and only if the error $E(P(x)) = 0$ for all polynomials $P(x)$ of degree $K = 0, 1, .., n$, but $E(P(x)) \neq 0$ for some polynomial $P(x)$ of degree $n+1$.

4.3)

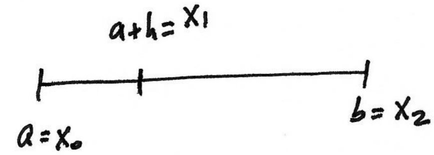
Probl. (16) If $h = \frac{b-a}{3}$, $x_0 = a$, $x_1 = a+h$, $x_2 = b$.

Find degree of precision of quad. formula

$$\int_a^b f(x)dx = \frac{9}{4}h\, f(x_1) + \frac{3}{4}h\, f(x_2)$$

$$= \frac{h}{4}\left[9f(x_1) + 3f(x_2)\right].$$

Answer :



a) For $P(x) \equiv 1$ (degree zero).

i) $\int_a^b dx = b-a$ ✓

ii) $\frac{h}{4}\left[9f(x_1)+3f(x_2)\right] = \frac{h}{4}(9+3) = 3h = \cancel{3}\left(\frac{b-a}{\cancel{3}}\right) = b-a.$ ✓

b) For $P(x) \equiv x$ (degree one)

i) $\int_a^b x\,dx = \frac{x^2}{2} = \frac{b^2-a^2}{2}$ ✓

ii) $\frac{h}{4}\left[9x_1 + 3x_2\right] = \frac{h}{4}\left[9(a+h) + 3b\right] =$

$= \frac{h}{4}\left[9a + 9\left(\frac{b-a}{3}\right) + 3b\right] = \frac{h}{4}\left[9a + 3b - 3a + 3b\right] =$

$= \frac{h}{4}\left[6a + 6b\right] = \frac{b-a}{12}(6a+6b)$

$= \frac{b-a}{2}(a+b) = \frac{b^2-a^2}{2}$ ✓

c) Try now with $P(x) \equiv x^2$.

c) For $P(x) = x^2$.

iii) $$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3 - a^3}{3}.$$

$$= \frac{(3h+a)^3 - a^3}{3}$$

Using the quad. formula:

$$\frac{\cancel{a^3} + 3(3h)^2 a + 3(3h)a^2 + 27h^3 - \cancel{a^3}}{3}$$

$$\int_a^b x^2\,dx \simeq \frac{h}{4}\left[9x_1^2 + 3x_2^2\right]$$

$$= \frac{27h^3 + 27ah^2}{3} + \frac{9a^2h}{3}$$

$$= \frac{h}{4}\left[9(a+h)^2 + 3b^2\right]$$

Same!

$$= \frac{1}{4}\left[9h(a+h)^2 + 3hb^2\right]$$

$$= \frac{1}{4}\left[9(a^2h + 2ah^2 + h^3) + 3h(3h+a)^2\right]$$

$$= \frac{1}{4}\left[9(a^2h + 2ah^2 + h^3) + \cancel{9h\,3a}\right.$$

$$\left. + 3h(9h^2 + 6ah + a^2)\right]$$

$$= \frac{1}{4}\left[36h^3 + 36ah^2 + 12a^2h\right]$$

$$= 9h^3 + 9ah^2 + 3a^2h.$$

d) Try with $P(x) = x^3$. (according to book it should also work)